Dynamic Optimization of Arbitrary 2D Structures Using Ground Structure Method with High-Order Composite Implicit Time Integration

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Abstract

This paper presents an implementation for ground structure-based topology optimization of arbitrary 2D structures subjected to dynamic load. Topology optimization of the structure is achieved by employing the Bi-directional Evolutionary Structural Optimization (BESO) algorithm, which aims to reduce the structural weight by using strain energy as the basis for the removal and addition of elements. To solve the structural dynamic problem, a high-order composite implicit time integration scheme is employed, utilizing the same effective stiffness matrix across all sub-steps to reduce computational costs. The effectiveness of the proposed method is validated through numerical examples.

Keywords: Ground structure method, Structural dynamics, Composite time integration, Bi-Directional Evolutionary Structural Optimization

1. Introduction

The ground structure method proposed by Dorn et al. [1] is a widely used approach in the topology optimization of trusses, which discretizes the domain into a network of interconnected potential members and optimizes their connectivity to achieve an optimal structural layout that minimizes the objective function and satisfies crucial constraints.

Ground structure-based approaches provide a flexible framework for optimizing structures, allowing for the efficient exploration of diverse topological configurations. Zegard and Paulino [2, 3] developed the computer implementation named GRAND and GRAND3, extending the ground structure method for arbitrary domains for two- and three-dimensional truss topology optimization, respectively. The growing ground structure method for topology optimization of trusses proposed by Hagishita and Ohsaki [4] optimizes the initial ground structure using growth strategies to determine the expansion and reduction of bars so that the optimal solution for the given set of nodes is generated.

However, most truss topology optimization studies are focused on finding the optimal solution for structures under static loads, even though structural systems are often subjected to dynamic loads arising from natural and human-made sources, such as earthquakes, wind, traffic, and machinery vibrations.

The effect of dynamic loads is crucial in civil engineering design. If not properly accounted for, these effects can lead to excessive deformation, structural damage, or even catastrophic failure.

To ensure structural stability, safety and serviceability of the optimized truss subjected to dynamic loading, this paper integrates the bi-directional evolutionary structural optimization (BESO) based on the ground structure method with high-order composite implicit time integration schemes. The high-order composite implicit time integration method is highly advantageous for stiff problem analyses and large-scale problems [5], which is beneficial for ground structure-based optimization.

2. Problem formulations

2.1 Ground structure generation

The ground structure method originally introduced by Dorn et al. [1] is a fundamental approach in structural optimization, providing a systematic framework for defining potential structural layouts. By discretizing a design space into a network of nodes and potential connecting members, the ground structure method will work efficiently when the problem is translated to linear algebra and matrix operations [2].

Ground structure-based topology optimization is extensively applied in civil engineering to optimize bridges, high-rise structures, and lightweight truss frameworks under static and dynamic loading conditions. This method generates a highly connected network of potential truss members based on predefined domain criteria and boundary conditions as shown in Fig. 1.

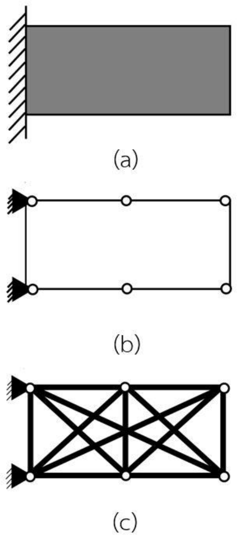


Fig. 1 The ground structure generating process: (a) the design domain, (b) nodes defined to discretize the domain, and (c) interconnected members.

This study adopts the ground structure analysis and design (GRAND) methodology developed by Zegard and Paulino [2] to generate a ground structure. The design domain, nodal coordinates in the x and y axes defining the geometric layout, support conditions, and applied external loads are input to create an element connectivity vector, representing potential truss members between nodes acting as a basis structure for an optimization algorithm to remove non-optimal elements selectively.

The interconnectedness of potential members can be adjusted by changing the connectivity level. If the connectivity level is sufficiently high, the ground structure generation algorithm will interconnect all nodes, referred to as a full-level ground structure. Level 1 connectivity will generate members between every node that belongs to the same element in the base mesh, considered neighboring nodes. Level 2 connectivity will connect members up to the neighbors of the neighbors [2]. Fig. 2 illustrates the ground structure generation example with 7 elements and 12 nodes in base mesh, which results in the ground structure for level 1 and 2 connectivity [2].

As the level of connectivity increases, the likelihood of overlapping truss members (members connecting collinear nodes) also rises as shown in Fig. 3. The collinearity check is provided in GRAND to detect overlapping members and delete them in the generating process using the directional cosines of members with predefined collinearity tolerance as a removing criterion [2].

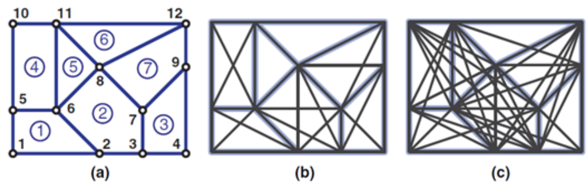


Fig. 2 Ground structure of 7 elements base mesh: (a) result of level 1 connectivity and (b) result of level 2 connectivity [2].

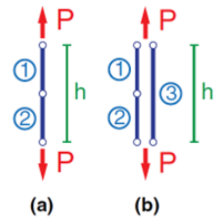


Fig. 3 Overlapping members example [2].

2.2 Structural dynamic problem solving

Structural dynamics deals with the behavior of structures subjected to time-dependent external excitations, such as earthquakes, wind loads, and machinery vibrations, unlike static analysis, where loads are assumed to be constant. Accurately predicting a structure's dynamic response ensures stability, safety, and serviceability in civil engineering applications.

Solving structural dynamic problems involves formulating equations of motion that describe how structures respond to external forces over time, as stated below:

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t)+\boldsymbol{C}\dot{\boldsymbol{u}}(t)+\boldsymbol{K}\boldsymbol{u}(t)=\boldsymbol{F}(t), \quad (1)$$

where $\boldsymbol{F}$ is the external excitation force vector, and $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ denote the mass, damping, and stiffness matrices, respectively. $\ddot{\boldsymbol{u}}$, $\dot{\boldsymbol{u}}$ and $\boldsymbol{u}$ represent acceleration, velocity, and displacement vectors, respectively. A dot (•) above the symbol denotes the derivative with respect to the time t.

Various numerical methods, such as modal analysis, time integration techniques, and frequency response analysis, are employed to analyze the dynamic behavior of structures.

In this paper, dynamic analysis is performed by high-order composite implicit time integration schemes developed by Chongmin Song and Xiaoran Zhang [5]. This method is advantageous in the structural analysis of truss generating from GRAND because of the use of the same effective stiffness in all sub-steps, reducing computational costs for systems with large degrees of freedom.

The time step size at the time step n $(n = 1, 2, \ldots)$ over the simulation time is denoted as:

$$\Delta t = t_n - t_{n-1}, \tag{2}$$

and the time within the time step n can be determined by:

$$t(s) = t_{n-1} + s\Delta t, \quad 0 \le s \le 1, \tag{3}$$

where s is a dimensionless time variable with $t(s=0) = t_{n-1}$ and $t(s=1) = t_n$ at the beginning and the end of the time step, respectively.

The derivative of the velocity and acceleration vectors with respect to the dimensionless time, denoting by a circle (◦) above the symbol, within the time step $t_{n-1} \le t \le t_n$ can be written as:

$$\dot{\boldsymbol{u}} = \frac{1}{\Delta t}\frac{d\boldsymbol{u}}{ds} = \frac{1}{\Delta t}\overset{\circ}{\boldsymbol{u}}, \tag{4}$$

$$\ddot{\boldsymbol{u}} = \frac{1}{\Delta t^2}\frac{d^2\boldsymbol{u}}{ds^2} = \frac{1}{\Delta t^2}\overset{\circ\circ}{\boldsymbol{u}}, \tag{5}$$

leads to the equation of motion in the dimensionless time as follows:

$$\boldsymbol{M}\overset{\circ\circ}{\boldsymbol{u}} + \Delta t\boldsymbol{C}\overset{\circ}{\boldsymbol{u}} + \Delta t^2\boldsymbol{K}\boldsymbol{u} = \Delta t^2\boldsymbol{f}. \tag{6}$$

By introducing the state-space vector:

$$\boldsymbol{z} = \begin{Bmatrix} \overset{\circ}{\boldsymbol{u}} \\ \boldsymbol{u} \end{Bmatrix}, \tag{7}$$

equation (6) can be transformed into a system of first-order ordinary differential equations (ODEs) as shown in equation (8):

$$\overset{\circ}{\boldsymbol{z}} \equiv \frac{d\boldsymbol{z}}{ds} = \boldsymbol{A}\boldsymbol{z} + \boldsymbol{F}, \tag{8}$$

where $\boldsymbol{A}$ denotes the constant coefficient matrix and $\boldsymbol{F}$ is the non-homogeneous term defined as:

$$\boldsymbol{A} = \begin{pmatrix} -\Delta t\boldsymbol{M}^{-1}\boldsymbol{C} & -\Delta t^2\boldsymbol{M}^{-1}\boldsymbol{K} \\ \boldsymbol{I} & 0 \end{pmatrix}, \tag{9}$$

$$\boldsymbol{F} = \begin{Bmatrix} \Delta t^2\boldsymbol{M}^{-1}\boldsymbol{f} \\ 0 \end{Bmatrix}. \tag{10}$$

The solution at the intermediate time point of the time step n is only used to calculate the solution at the end of the time step, which can be expressed using the matrix exponential function as:

$$\boldsymbol{z} = e^{\boldsymbol{A}s}\boldsymbol{z}_{n-1} + e^{\boldsymbol{A}s}\int_0^s e^{-\boldsymbol{A}\tau}\boldsymbol{F}(\tau)\,d\tau. \tag{11}$$

To simply generalize the formulation, the Padé expansion is used to replace the matrix exponential function, resulting in equation (12):

$$\boldsymbol{Q}\boldsymbol{z}_n = \boldsymbol{P}\boldsymbol{z}_{n-1} + \sum_{k=0}^{p_f}\boldsymbol{C}_k\boldsymbol{F}_{mn}^{(k)}, \tag{12}$$

where $\boldsymbol{F}$ is the force vector approximated by a polynomial expansion of order p_f, $\boldsymbol{P}$ and $\boldsymbol{Q}$ are polynomials of matrix $\boldsymbol{A}$ expressed as:

$$\boldsymbol{P} = \sum_{i=0}^{N_p} p_i\boldsymbol{A}^i = p_0\boldsymbol{I} + p_1\boldsymbol{A} + \ldots + p_{N_p}\boldsymbol{A}^{N_p}, \tag{13}$$

$$\boldsymbol{Q} = \sum_{i=0}^{Nq} q_i\boldsymbol{A}^i = q_0\boldsymbol{I} + q_1\boldsymbol{A} + \ldots + q_{N_q}\boldsymbol{A}^{Nq}, \tag{14}$$

and the matrices $\boldsymbol{C}_k$ are determined as:

$$\boldsymbol{C}_k = \boldsymbol{A}^{-1}\left(k\boldsymbol{C}_{k-1} + (-0.5)^k\left(\boldsymbol{P} - (-1)^k\boldsymbol{Q}\right)\right), \tag{15}$$

where $k = 1, 2, \ldots, p_f$.

After solving equation (12), the displacement vector $\boldsymbol{u}_n$ is obtained in the upper half of the vector $\boldsymbol{z}_n$, and the velocity $\dot{\boldsymbol{u}}_n$ can be obtained using the derivative with respect to the dimensionless time $\mathring{\boldsymbol{u}}_n$ in the lower half of $\boldsymbol{z}_n$.

There are two schemes available in this approach. An M-scheme allows the user to specify the spectral radius at the high-frequency limit to control the amount of numerical dissipation. On the other hand, an (M + 1)-scheme has a fixed amount of numerical dissipation built into the formulation [5]. As for the dynamic analysis of trusses in this paper, the M-scheme is chosen with the number of sub-steps p in a time step equals to 1 and the spectral radius ρ_∞ is 0.

2.3 Bi-Evolutionary Structural Optimization

2.3.1 BESO concepts

Topology optimization is particularly powerful as it automatically creates optimal structural designs that fulfill designated criteria within the design domains, allowing the structure to evolve by systematically removing inefficient material to ensure that only the necessary structural elements remain [6].

Various algorithms have been developed for topology optimization, including the Solid Isotropic Material with Penalization (SIMP) method [7, 8], Level Set methods, and Evolutionary Structural Optimization (ESO) approaches. Among these, the Bi-directional Evolutionary Structural Optimization (BESO) algorithm has gained significant attention due to its efficiency, adaptability, and effectiveness in structural optimization problems, especially in truss structures.

The BESO algorithm was first introduced by Xie and Steven [9] as an enhancement of the original ESO method. It is an iterative optimization method that refines a structure by allowing both element removal and addition of elements, leading to more flexible and efficient designs. This method is particularly effective in ground structure-based topology optimization, where a dense network of potential truss members (ground structure) is optimized by eliminating unnecessary elements while reinforcing critical load paths.

The BESO algorithm incorporates the concept of the ESO method, which eliminates the inefficient element, with the additive evolutionary structural optimization (AESO) algorithm, which, in contrast, adds elements to prevent some elements from being overly stressed. BESO effectively merges top-down and bottom-up optimization, clarifying ESO's common query about the return of removed elements, and its dual capability to add or eliminate facilitates a wider and deeper search in the design domain, which increases the chances of finding global minima while also significantly cutting down on computational time [10].

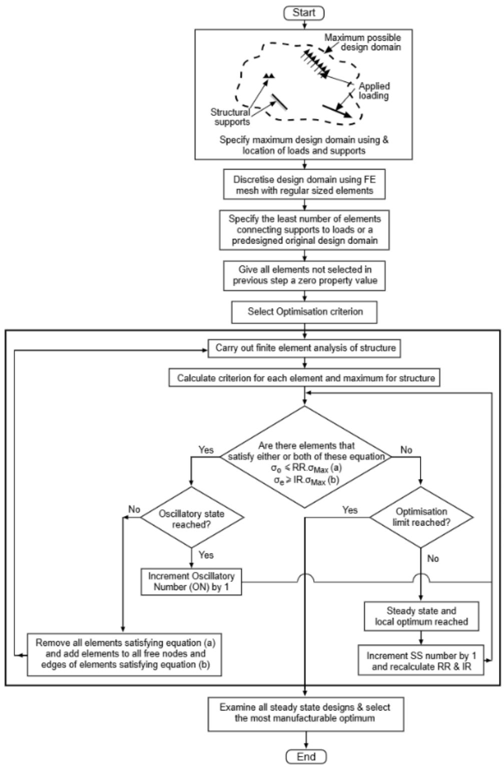


Fig. 4 BESO algorithm [10].

2.3.2 BESO parameters

After performing a dynamic analysis, the BESO parameters should be defined before entering the optimization process. Since the objective of this optimization is the minimum volume of the truss structure, a reduction of volume is expected, and it will be iteratively reduced to the target volume fraction $volfrac$ determined as the ratio of the target volume to the initial volume.

Even though this algorithm consists of iterations to perform, without a limited amount of change in each iteration, the target volume can be reached in a non-evolutionary manner. Therefore, the structure volume will be reduced by some amount of fraction called the elimination ratio ER which is the ratio of the final volume at the end of an iteration to the initial volume of that iteration that lies between zero and one.

The optimization process continues to iterate in an outer loop until the stopping criterion, assessed by the trend in compliance values $Comp$, is satisfied:

$$\left| \frac{\sum_{m=5}^{m=9} Comp_{n-m} - \sum_{m=1}^{m=4} Comp_{n-m}}{\sum_{m=1}^{m=4} Comp_{n-m}} \right| < tol \;, \tag{16}$$

where n is the current iteration number, m is an integer that denotes the difference between the current iteration and the iteration number used to calculate the compliance trend, and tol is the predefined tolerance.

This criterion requires the compliance of each iteration, which is derived from the summation of the remaining elements' compliances:

$$Comp_i = \sum_{e=1}^{N_e} \left(x_e comp_e \right) , \tag{17}$$

where i, N_e and x_e denote the iteration number, the number of elements within the design domain and the design variables of element e (see section 2.3.3). The element compliances $comp_e$ are calculated using the maximum element compliance and its 2 adjacent values, as shown below:

$$comp_e = comp_e^{(t_e)} + comp_e^{(t_e+1)} + comp_e^{(t_e-1)} , \tag{18}$$

where t_e is the time step number that has maximum compliance value. The compliance value at the time step t is defined as:

$$comp_e^{(t)} = \frac{1}{2} \boldsymbol{u}_e^{T(t)} \boldsymbol{k}_e \boldsymbol{u}_e^{(t)} , \tag{19}$$

where $\boldsymbol{u}_e$ is the displacement of element e and $\boldsymbol{k}_e$ is the element stiffness matrix of element e in global coordinate.

2.3.3 Element addition and deletion

When dealing with element addition and deletion, design variables x are assigned to elements within the ground structure domain to determine whether to keep (i.e. $x = 1$) or eliminate (i.e. $x = 0$) them. This process is iteratively executed through inner-loop iterations.

A sensitivity number α is employed to define the design variable of elements. Each element in the structure is assigned a sensitivity number to quantify its contribution to the optimization process. Elements with greater sensitivity contribute more significantly to the overall structure, demonstrating a stronger influence on the structure's behavior when their design variables are modified.

The sensitivity values of element e can be identified as follows [11]:

$$\alpha_e = x_e comp_e . \tag{20}$$

and they are then refined using the filter scheme to prevent the optimization process from being affected by checkerboard patterns:

$$\alpha_e = \frac{\sum_{j=1}^{N_e} w_{ej} \alpha_j}{\sum_{j=1}^{N_e} w_{ej}} \tag{21}$$

where j represents elements other than the element of interest e and w_{ej} denotes the weight factor.

Based on the prescribed filter radius $r_{\min}$ and the distance Δ_{ej} between the centers of elements e and j, the weight factor is calculated as:

$$w_{ej} = \max\left(0, r_{\min} - \Delta_{ej}\right) . \tag{22}$$

The defined criterion governs the elimination and addition processes:

$$\frac{L_{\max} + L_{\min}}{L_{\max}} < 10^{-5} \tag{23}$$

where $L_{\max}$, $L_{\min}$ are the maximum and the minimum sensitivity value of element within the domain, respectively.

To assign the design variables, a criterion in equation (24) is prescribed:

$$x_e = \begin{cases} 1 & \text{when } \alpha_e > \alpha_{th} \\ 0 & \text{when } \alpha_e \le \alpha_{th} \end{cases} \tag{24}$$

where α_{th} is the midrange between $L_{\max}$ and $L_{\min}$. After the element design variables are all assigned, if the current volume ratio is greater than the iteration's target volume ratio, the $L_{\min}$ value for the following iteration shall be set to the current α_{th} value to reduce the volume by preserving elements with higher sensitivity. On the other hand, if the current volume ratio is lower than the iteration's target volume ratio, the $L_{\max}$ value for the following iteration shall be set to the current α_{th} value to preserve more elements. This algorithm iteratively refines the volume until it converges to the target volume, achieving convergence in both $L_{\max}$ and $L_{\min}$.

Once the equation (23) is satisfied, the outer loop begins again, starting from the dynamic analysis procedure, targeting the updated volume, which is decreased from the prior value by ER.

2.3.4 Post-processing

The outer loop continues until the $volfrac$ and the criterion in equation (16) is met. However, there might be overstressed and unstable (hair-like) elements. Therefore, algorithms to detect and remove them are needed. This stage consists of 2 looping algorithms.

The first one is for handling overstressed elements. A dynamic analysis is conducted to evaluate the elements' stress. If any element is found to be under excessive stress, its design variable will be adjusted to zero. This process will repeat until all elements are under the stress limit.

The second loop is to deal with unstable elements. Since unstable elements possess a protruding end, nodes located at these ends will have one element linked to them. By identifying nodes with only a single element connected, we can pinpoint the unstable element and assign its design variable to zero. This procedure will continue until no unstable elements remain.

3. Numerical examples

Table 1 Input parameters used for dynamic analysis process.

Parameter	value
Composite time integration scheme, $scheme$	M
Number of sub-steps, p	1
Spectral radius, ρ_∞	0
Damping matrix, $\boldsymbol{C}$	0
Initial displacement, $\boldsymbol{u}^{(0)}$	0
Initial velocity, $\dot{\boldsymbol{u}}^{(0)}$	0

Structures in the following examples are made of steel with Young's modulus $E = 200\ \text{GPa}$ and mass density $\rho = 7{,}850\ \text{kg/m}^3$. To perform dynamic analysis, input parameters shown in Table 1 are used.

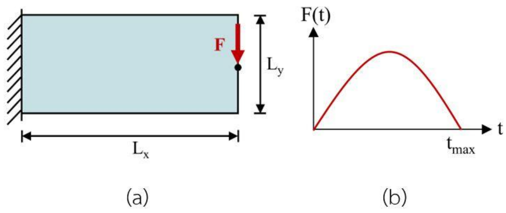


Fig. 5 Rectangular cantilever beam: (a) design domain and boundary conditions and (b) applied load.

3.1 Rectangular cantilever beam

A rectangular cantilever beam shown in Fig. 5 has the domain size $L_x \times L_y = 3\times 1\ \text{m}$ subjected to a half-cycle sine loading defined as $F(t) = \sin(\pi t / t_{\max})\ \text{kN}$ at the middle-central point of the free edge over the duration $t_{\max} = 0.1\ s$ with a time step size $\Delta t = 0.001\ s$. The domain is discretized into 30 x 10 base meshes with level 3 connectivity using GRAND. The BESO parameters, including the elimination ratio ER, the target volume fraction $volfrac$ and filter radius $r_{\min}$ are set to 0.02, 0.2 and 0.1, respectively. Figure 6 shows the base mesh and the optimized result. The evolution history is presented in Fig. 7.

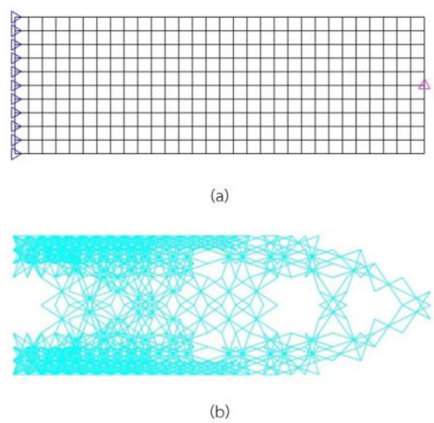


Fig. 6 Rectangular cantilever beam: (a) base meshes of discretized structure and (b) solution obtained.

The half-cycle sine load only differs in magnitude but maintains the downward direction. It affects the cantilever beam in the same way as a static load does. So, a similar result is expected when using this methodology to optimize the same cantilever beam with the same set of parameters, but subjected to a static load. Figure 8 shows the result that supports the assumption. Although the final volume fraction of the dynamic load case, which is 0.249, is higher than 0.224 in the static load case, the final structures are similar.

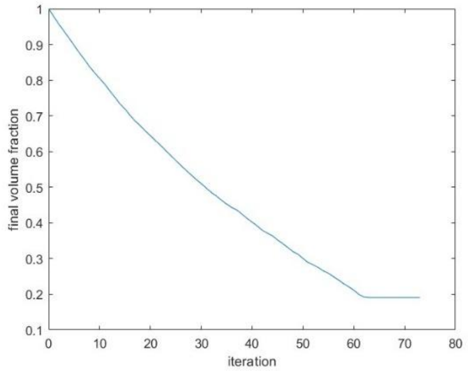


Fig. 7 Evolution history of the rectangular cantilever beam.

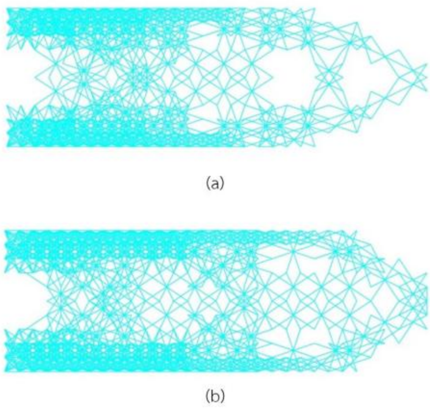


Fig. 8 Optimized rectangular cantilever beams: (a) dynamic load case and (b) static load case.

3.2 Cantilever beam with circular support

The design domain is a cantilever beam with a circular support loaded vertically with a half-cycle sine load applied at the center of the free edge as shown in Fig. 9. The beam has dimensions $B = 5\,\mathrm{m}$, $H = 4\,\mathrm{m}$ and $r = 1\,\mathrm{m}$, and is discretized into 300 polygonal meshes with level 3 connectivity.

Figure 10 presents the polygonal meshes of the discretized structure and the optimized design obtained for duration $t_{\max} = 0.1\,s$ with a time step size $\Delta t = 0.001\,s$ subjected to the applied load $\boldsymbol{F}(t) = \sin(\pi t / t_{\max})\ \mathrm{kN}$. In the optimization process, the elimination ratio ER is set to 0.02, the target volume fraction $volfrac$ is 0.1, and the filter radius $r_{\min}$ is 0.1. The evolution history is displayed in Fig. 11.

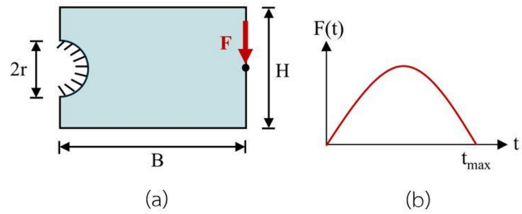


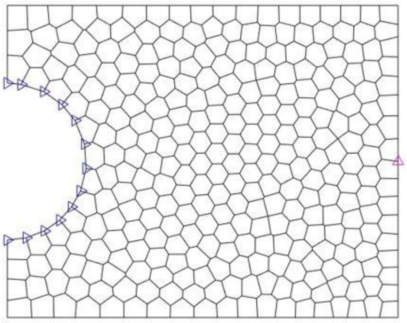
Fig. 9 Cantilever beam with circular support: (a) design domain and boundary conditions and (b) applied load.

Although the applied load used in this example changes over time, only its magnitude changes. The direction of the load $\boldsymbol{F}(t)$ is only downward. Thus, we expect the optimized result when switching from dynamic loading to static loading, using the same methodology and parameters, to not be much different. Fig. 12 proves the assumption, showing that the optimized structure under dynamic and static load differs slightly. The final volume fraction in the dynamic load case is 0.63 and 0.066 in the static load case.

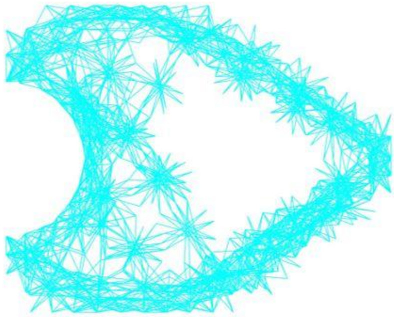
4. Conclusions

This paper presents a computer implementation methodology for ground structure-based topology optimization of truss under dynamic loading using ground structure analysis and design (GRAND) with high-order composite implicit time integration schemes to ensure structural stability and performance. This methodology is employed in the study to minimize the weight of the structure.

GRAND enables unstructured mesh generation, which can offer a more accurate model, i.e. cantilever beam with circular support, for further optimization process. High-order composite implicit time integration scheme is computationally efficient for optimizing ground structures, especially ones with highly connected members generated from GRAND, which has large degrees of freedom. Furthermore, this method reduces computational costs using the same effective stiffness matrix in all sub-steps.



(a)



(b)

Fig. 10 Cantilever beam with circular support: (a) polygonal meshes of discretized structure and (b) solution obtained.

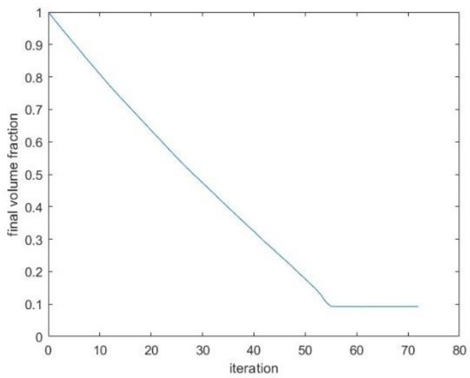
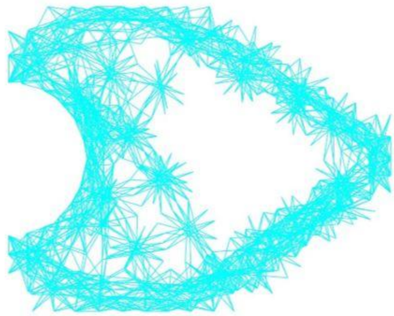


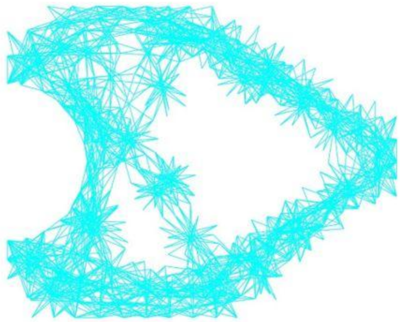
Fig. 11 Evolution history of the cantilever beam with circular support.

The bi-directional evolutionary structural optimization (BESO) algorithm is utilized for optimization, offering computational efficiency and robustness.

The proposed methodology effectively achieves the desired volume through a systematic reduction of the volume fraction across iterations, converging toward the target, while also ensuring the integrity and safety of the results.



(a)



(b)

Fig. 12 Optimized cantilever beams with circular support: (a) a dynamic load case and (b) a static load case.

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