Topology Optimization Using the SBFE-BESO Approach for Composite Structure Design

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Abstract

This paper presents a topology optimization method that integrates the image-based Scaled Boundary Finite Element (SBFE) method with the Bi-directional Evolutionary Structural Optimization (BESO) algorithm. The approach targets composite structures composed of multiple materials, each exhibiting distinct mechanical properties under tension and compression. The structure is partitioned into tension and compression regions using the first invariant of the stress tensor, derived from the sum of the diagonal (or principal) stresses. Materials optimally suited for each stress state are then selectively assigned to their corresponding regions. Engineering structures are used to demonstrate the method's effectiveness. Results indicate that the proposed approach enhances structural performance, improves material utilization compared to single-material optimization, and reduces overall material volume, contributing to cost efficiency.

Keywords: Multi-material topology optimization, Topology optimization, Bi-directional evolutionary structural optimization (BESO), Multi-phase material design, Material cost efficiency, Stress-based distribution, MATLAB, Engineering Institute of Thailand, National Convention on Civil Engineering

1. Introduction

In modern structural design, integrating architectural aesthetics with computational engineering analysis is essential for advancing construction techniques. Structural efficiency has evolved beyond traditional goals such as stability and strength to include lightweight, cost-effective, and visually appealing solutions that minimize resource use. As illustrated in Fig. 1, contemporary designs emphasize not only performance and lightness but also elegance, often inspired by natural forms.

Achieving harmony between form and function requires a holistic design approach from the early stages of development. Topology optimization has emerged as a powerful technique for creating structurally efficient and geometrically complex designs. Among the various methods, the Bi-directional Evolutionary Structural Optimization (BESO) technique [1, 2] stands out for its ability to iteratively remove and add material to enhance performance. While most research has focused on single-material systems, this research addresses the more complex challenge of optimizing composite structures composed of materials with distinct tensile and compressive properties for example, placing steel in tensile zones and concrete in compressive ones to maximize material efficiency.

Fig. 1 Innovative architectural design example based on the BESO technique: Qatar National Convention Center.

A key challenge in topology optimization is balancing mesh resolution with computational cost. High-resolution meshes improve accuracy in Finite Element Analysis (FEA) but significantly increase computational demands. Although adaptive mesh (AM) techniques offer a partial solution, they introduce issues such as hanging nodes and longer computation times. To overcome these limitations, this research adopts an image-based Scaled Boundary Finite Element (SBFE) method integrated with the BESO technique [3, 4]. The proposed SBFE-BESO approach employs a quadtree-based meshing strategy (i.e., Fig. 2), enabling flexible boundary definition and improved precision.

In particular, this research extends the SBFE-BESO framework to multi-material composite structures, especially those exhibiting distinct mechanical behaviors under tension and compression. This integration addresses a critical research gap, as few existing studies have combined adaptive meshing techniques with multi-material topology optimization in such complex composite scenarios.

This method enhances both computational efficiency and structural accuracy, making it a robust tool for multi-material topology optimization.

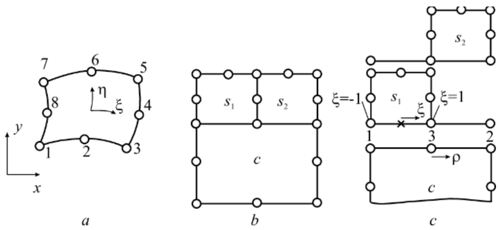

Fig. 2 A quadtree mesh of S-elements.

2. Image-based SBFE-BESO mesh generation

2.1 Introduction

An advanced automatic meshing technique was introduced by [5], which combines the Bi-directional Evolutionary Structural Optimization (BESO) method with the Scaled Boundary Finite Element (SBFE) method. This technique employs 2D quadtree meshing based on grayscale digital images in Standard Tessellation Language (STL) format. These images serve as the geometric basis for mesh generation, enabling the transformation of design domains into analysis-ready models.

A key advantage of this method lies in its ability to manage hanging nodes and create hierarchical mesh structures using convolution filtering. The filter adjusts grayscale intensity between solid and void regions, allowing for multi-level refinement and effectively reducing the number of degrees of freedom (DOFs) in the analysis.

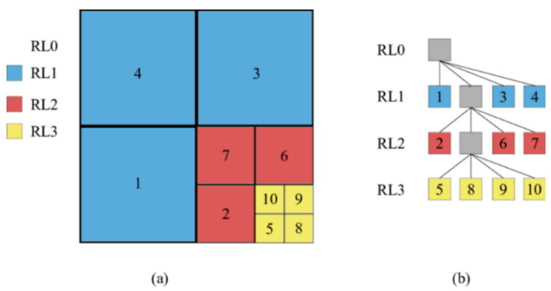

Fig. 3 Quadtree scheme in a 2D view: (a) Quadtree mesh representation and (b) Tree diagram of quadtree element.

2.2 Overview of Image-Based Quadtree Mesh Generation

Quadtree meshing is a hierarchical data partitioning technique used to divide a domain into four equal-sized quadrants or regions. Each quadrant can be further subdivided into smaller quadrants, known as “children”, which have half the grid spacing of the preceding layer. The top-level quadrant is referred to as the “parent”.

As the hierarchy progresses to deeper levels, the mesh becomes increasingly refined and complex. As shown in Fig. 3(a), the subdivision of quadrants from layer 0 (RL0) to layer 3 (RL3) is presented, while Fig. 3(b) displays the corresponding tree diagram. In this diagram, uncolored regions are further subdivided. Starting from RL0 (the parent), it is divided into four RL1 quadrants (children), which are in turn subdivided into RL2, and subsequently RL3, with each layer representing a successive level of refinement.

2.3 Adaptive Mesh Convolution-Based Filter Technique

The adaptive mesh convolution filtering method proposed by [6, 7] combines adaptive meshing with convolution-based filtering to achieve high topological accuracy and computational efficiency. Adaptive meshing concentrates computational resources in critical regions of the image, while convolution filters are used to smooth, sharpen, or extract features from the image.

The convolution filter defines a design variable based on the grayscale image for the SBFE decomposition algorithm, which generates a hierarchical analysis-ready mesh. In this context, black regions in the image represent solid structures, while white regions denote voids.

Quadtree-based SBFE decomposition produces a refined, non-uniform mesh within the grayscale domain, thereby reducing the number of DOFs in the analysis. In MATLAB, the convolution process can be performed using the ‘conv2’ function for 2D spaces. The filter is applied with a specified radius of density in pixels, modifying the grayscale intensity of each pixel within the image matrix.

2.4 Bi-directional Evolutionary Structural Optimization (BESO)

Since the late 1980s, topology optimization methods have undergone significant development. In the 1990s, the Evolutionary Structural Optimization (ESO) method was first proposed by [8], inspired by natural evolutionary processes observed in structures such as bones and trees (Fig. 4). The ESO method uses stress levels as criteria to iteratively remove inefficient material, gradually forming an optimized structure.

However, because ESO only considers local stress values, it may prematurely eliminate material that could become important in later stages of optimization. Once removed, such material cannot be recovered, which limits the method's flexibility and often necessitates starting with a higher initial material volume.

To overcome these limitations, a more advanced approach so-called Bi-directional Evolutionary Structural Optimization (BESO) was introduced by [1]. BESO allows for both removal and reintroduction of material during optimization, enhancing adaptability. As shown in Fig. 5(a–f), during the initial iterations, elements are added to the structure (i.e., Fig. 5(b–c)), enabling evolution toward an optimal shape under applied loads. As the structure evolves, gaps may form (see, Fig. 5(d)), and the final optimized structure is achieved as presented in Fig. 5(f).

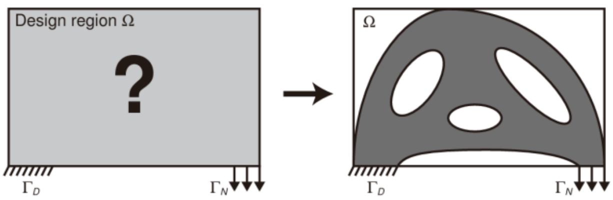

Fig. 4 Illustration of typical structural topology optimization.

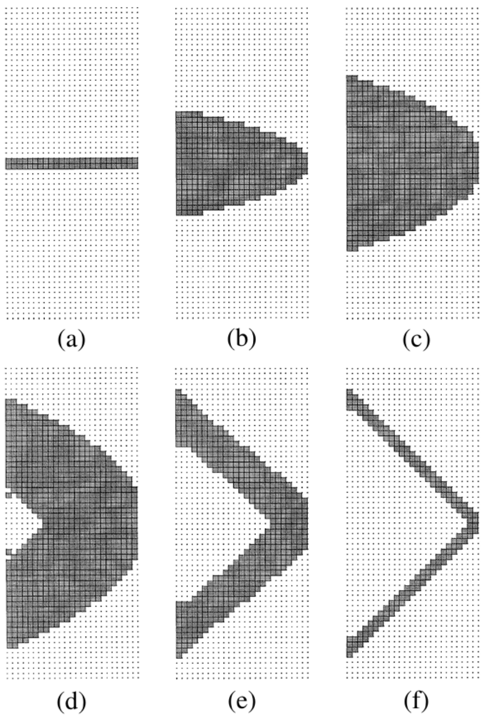

Fig. 5 The iterative of BESO design process for a two-bar frame [9].

2.5 Criterion of tension or compression

To determine the tensile or compressive state of an element in a structure, [10-12] proposed using the first invariant of stress tensors of the i-th element $\left(I_{1,i}\right)$ as a criterion. Where i represent the number of each element, and $\left(I_{1,i}\right)$ can be expressed as follows:

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33}, \tag{1}$$

where σ_{11} , σ_{22} and σ_{33} represent the maximum principal stress, second principal stress, and minimum principal stress, respectively. For a 2D problem, only the maximum and minimum principal stresses are necessary in the analysis.

$\left(I_{1,i}\right)$ is also three times the hydrostatic stress (or called volumetric stress). So $I_{1,i} \geq 0$, means the i-th element is volumetric expansion and can be considered under tension, while $I_{1,i} \leq 0$ means the volume of the i-th element is reduced, which can be regarded as being in compression. Fig. 6 shows a cantilever beam subjected to concentrated loads at the center-right, while Fig. 7 illustrates the tensioned and compressed material after using topology optimization, where the red area represents the tension zone and the blue area represents the compression zone.

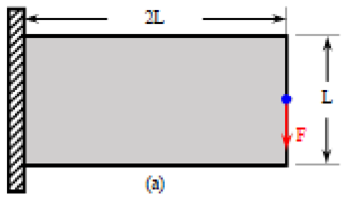

Fig. 6 Design domain, support and loading conditions.

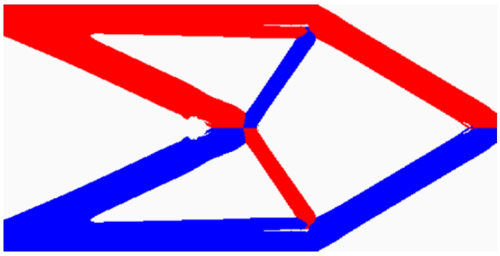

Fig. 7 Topology optimization result.

3. SOLUTION METHODOLOGY

3.1 Problem statements

In modern engineering applications, such as civil infrastructure, aerospace, and medical devices, lightweight and high-performance structures are essential. Traditional single-material topology optimization often fails to meet these performance needs due to limitations in material properties.

To overcome this, multi-material topology optimization is adopted, where materials are categorized into: Tensile group (strong in tension, e.g., steel):

$$E_1^t > E_2^t > \ldots > E_n^t, \quad (2)$$

and Compressive group (strong in compression, e.g., concrete):

$$E_1^c > E_2^c > \ldots > E_n^c, \quad (3)$$

each group contains n materials ordered by Young's modulus.

In a topology optimization problem that aims to maximize structural stiffness, the objective is to minimize the total strain energy of the structure while subject to a volume fraction constraint. This can be formulated mathematically as follows:

$$\text{Minimize: } c(\rho_t, \rho_c) = \frac{1}{2}\mathrm{F}^{\mathrm{T}}\mathrm{u}(\rho_t, \rho_c). \quad (4)$$

$$\text{Subject to: } K(\rho_t, \rho_c)u = F, \quad (5)$$

$$\begin{aligned} &V^* - \left(\sum_{i=1}^{N} V_i\rho_{t_i} + \sum_{i=1}^{N} V_i\rho_{c_i}\right) = 0,\\ &V_{t,j}^* - \left(\sum_{i=1}^{N} V_i\rho_{t_{ij}} + \sum_{i=1}^{j-1} V_{t,i}^*\right) = 0,\\ &V_{c,j}^* - \left(\sum_{i=1}^{N} V_i\rho_{c_{ij}} + \sum_{i=1}^{j-1} V_{c,i}^*\right) = 0,\\ &V_{t,j}^* = R_{t,j}V_t^*, \quad (6)\\ &V_{c,j}^* = R_{c,j}V_c^*,\\ &\rho_{t_{\min}} \le \rho_{t_{ij}} \le 1 \ (j = 1, 2, \ldots, n-1),\\ &\rho_{c_{\min}} \le \rho_{c_{ij}} \le 1 \ (j = 1, 2, \ldots, n-1), \end{aligned}$$

where $c(\rho_t, \rho_c)$ represents the total strain energy or mean compliance of the structure. $K(\rho_t, \rho_c)$ is the global stiffness matrix, and $u(\rho_t, \rho_c)$ is the displacement vector of the DOFs. V_i is the volume of the i-th element in the structure, and V^* is the predefined target volume fraction of the total material, $V_{t,j}^*$ and $V_{c,j}^*$ are the volumes of material in tension and compression for the j-th material, respectively. $\rho_{t_{ij}}$ and $\rho_{c_{ij}}$ are varibles of i-th materials representing the tensile or compressive state of j-th element. In the design of the material, the values of $\rho_{t_{ij}}$ and $\rho_{c_{ij}}$ which are 1 represent the presence of rigid materials under tension and compression, respectively, while the $\rho_{t_{\min}}$ and $\rho_{c_{\min}}$ are set to 0.001 and 0.002, respectively, to simulate void elements (to avoid singularity issues in the stiffness matrix). $R_{t,j}$ and $R_{c,j}$ are the ratios of the material volume associated with E_j^t or E_j^c to the total volume of material in tension or compression. This process does not predetermine the volume fraction of tensile or compressive regions before topology optimization process, instead, these values are computed during the optimization process according to Eq.(9) and (10), in each interation. The variables $\rho_{t_{ij}}$ and $\rho_{c_{ij}}$ are defined as Eq (7) and (8), respectively.

$$\rho_{t_{ij}} = \begin{cases} \rho_{t_{ij}} = 1 & \text{for } E > E_j^t, \text{and in tension} \\ \rho_{t_{ij}} = \rho_{t_{\min}} & \text{for } E \le E_{j+1}^t, \text{or in compression} \end{cases}, \quad (7)$$

$$\rho_{c_{ij}} = \begin{cases} \rho_{c_{ij}} = 1 & \text{for } E > E_j^c, \text{and in compression} \\ \rho_{c_{ij}} = \rho_{c_{\min}} & \text{for } E \le E_{j+1}^c, \text{or in tension} \end{cases}, \quad (8)$$

3.2 Material interpolation scheme

To define the material properties of each component in the topology optimization process for multiple materials, a material interpolation scheme is introduced. This scheme assists in optimizing the distribution of multiple materials while considering whether the element is in tension or compression. The interpolation scheme specifies the material property transitions between the j-th tensile material and the j+1-th tensile material or the j-th compressive material and the j+1-th compressive material. This material interpolation scheme ensures a smooth and efficient transition between materials of varying properties. The material interpolation equation are follows:

$$E(\rho_{t_{ij}}) = E_{j+1}^t + \rho_{t_{ij}}^p (E_j^t - E_{j+1}^t) \ (j = 1, 2, \ldots, n), \quad (9)$$

$$E(\rho_{c_{ij}}) = E_{j+1}^c + \rho_{c_{ij}}^p (E_j^c - E_{j+1}^c) \ (j = 1, 2, \ldots, n), \quad (10)$$

where $E(\rho_{t_{ij}})$ and $E(\rho_{c_{ij}})$ are the interpolated material properties for the material in tensile and compressive states, respectively, for the i-th element of the j-th material. p is the penalization factor, which controls the interpolation of material properties in the topology optimization process. In this research, it is set to a value of 3.

3.3 Sensitivity function

Sensitivity analysis identifies where to add or remove materials:

$$\alpha_{ij} = -\frac{1}{p}\frac{\partial c}{\partial \rho_{t_{ij}}} = \frac{1}{2}\rho_{t_{ij}}^{p-1}(u_i^T K_{t,i}^j u_i - u_i^T K_{t,i}^{j+1} u_i), \quad (11)$$

$$\beta_{ij} = -\frac{1}{p}\frac{\partial c}{\partial \rho_{c_{ij}}} = \frac{1}{2}\rho_{c_{ij}}^{p-1}(u_i^T K_{c,i}^j u_i - u_i^T K_{c,i}^{j+1} u_i). \quad (12)$$

To prevent mesh dependency and checkerboarding, filtering is applied:

$$\bar{\alpha}_i = \frac{\sum_{k=1}^{m}\omega(r_{ik})\alpha_i}{\sum_{k=1}^{m}\omega(r_{ik})}, \quad (13)$$

$$\bar{\beta}_i = \frac{\sum_{k=1}^{m}\omega(r_{ik})\beta_i}{\sum_{k=1}^{m}\omega(r_{ik})}, \quad (14)$$

where $\bar{\alpha}_i$ and $\bar{\beta}_i$ are the filtered sensitivity numbers for each element m in the tensile and compressive regions. $\omega(r_{ik})$ is the weight function:

$$\omega(r_{ik}) = \max(r_{\min} - r_{ik}, 0) . \tag{15}$$

3.4 Optimality Criteria

Before optimization begins, strong materials (M_1^T, M_1^C) and weak materials (M_2^T, M_2^C) are defined. Initially, all material volume is assigned to the strong materials.

The target volume of strong materials is reduced using the evolution rate (er):

$$V_s^k = V_s^{k-1} \times (1 - er), \tag{16}$$

the volume of weak materials increases accordingly:

$$V_w^k = 1 - V_s^k, \tag{17}$$

this gradual transition allows the optimizer to balance strength and material efficiency by shifting from strong to weak materials. To ensure stability and convergence, the following criterion is used:

$$\frac{\left|\sum_{p=1}^{Q}(C_{\kappa-p+1} - C_{\kappa-Q-p+1})\right|}{\sum_{p=1}^{Q} C_{\kappa-p+1}} \leq \tau , \tag{18}$$

Where τ is the error value, set to 0.001%, and Q is an integer, set to 5.

3.5 Proposed optimization algorithm

1. Create 16 Quadtree elements, as outlined by [2] This step sets up the initial mesh for subsequent analysis.

2.Define the following setting and parameters: boundary conditions and loading (Specify the supports and applied forces), young's modulus for each material and Poisson's ratio (set to 3 in this case), evolution rate, penalty factor, sensitivity filter radius, tolerance value and maximum number of iterations.

3.Set total target volume (V^*) , target volume for tensile materials $V_{t,j}^*$ and target volume for compressive materials $V_{c,j}^*$ by ratios of the material volume($R_{t,j}$ and $R_{c,j}$) And calculate the volume fractions for each material type. Use Eq. (16) and (17) and then repeat if the target volume is still greater than the specified target volume in the case of strong materials, and when the specified target is still less than the specified target volume in the case of weak materials.

4.Analyze the scale boundary finite element. The domain will be divided by the quadtree algorithm, and calculate the displacement field, strain field and stress field.

5. Calculate the first invariant of stress tensor (I_1) according to Eq. (1) and the sensitivity number in each S-element according to Eq. (11) and (12).

6.Update the design variables $(\rho_{t_{ij}}, \rho_{c_{ij}})$ based on the sensitivity values. If the sensitivity value is below the threshold, set the density value to the minimum (void). If the sensitivity value exceeds the threshold, set the density value to 1 (solid).

7. Check the convergence criterion. If the criteria are met the specified criteria, stop the process. If not, repeat Steps 1 to 6 until the convergence criteria are satisfied.

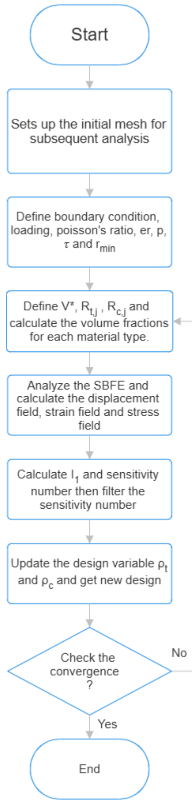

Fig. 8 Flow chart of the process.

4. Design Examples

This section presents an example of an optimal structural design for a simple 2D beam. The boundary conditions are defined such that the lower-left end of the beam is pinned, while the lower-right end is roller, shown in Fig. 9.

A concentrated load is applied at the center of the top edge of the beam. The magnitude of the load is set to 1 unit to observe the structural behavior. The Automatic Digital Image-

Based SBFE-BESO method is employed to compare compliance values and the material volume usage.

Steel is specifically considered due to its high Young's modulus and relatively high cost. In this research, materials are color-coded as follows: Steel (blue), Iron (gray), Ultra-High Performance Concrete (UHPC) (red), and Concrete C30 (yellow). The comparison includes single-material, two-material, and four-material cases, as illustrated in Fig. 9.

Generally, the total material volume fraction is set to 0.5, and the evolution rate (ER) is set to 5%. The radius for density filtering (rden) is 8. The optimization process terminates when the change in design is less than 0.0001 or the number of iterations exceeds 200. The implementation is carried out in MATLAB, using a 1280×256 pixel image as the input for element generation. The Young's modulus and volume fraction of each material are defined and summarized in Table 1.

Table 1 Material properties and volume fractions

		Materials volume fractions			
		steel	Iron	UHPC	Concrete
	Price(Bath/t)	25000	20000	15000	2200
	Young's modulus(Gpa)	200	100	55	28
	Poisson's ratio	0.2	0.2	0.2	0.2
	Density(kg/m³)	7850	6600	2300	2360
Design 1	Steel	0.5	-	-	-
Design 2	Steel+UHPC	0.25	-	0.25	-
Design 3	Steel+Iron+UHCP+Concrete	0.15	0.1	0.15	0.1

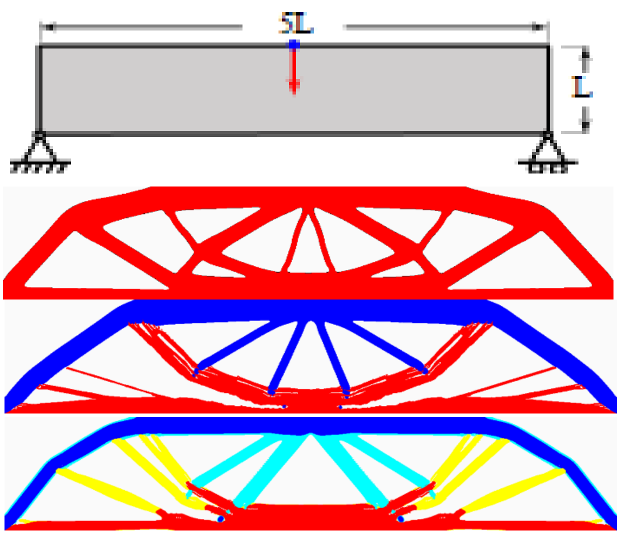

Fig. 9 Problem statement and the results of topology optimization, a) The optimal topology of 1 material, b) The optimal topology of 2 materials, and c) The optimal topology of 4 materials.

Fig. 9 illustrates the results of the topology optimization process. In all three cases, the red material represents the same material, designated as the primary (strong) material with high tensile mechanical properties. The blue material represents the primary (strong) material under compressive mechanical conditions. For the secondary (weaker) materials, yellow is assigned to tensile regions and cyan to compressive regions.

From Fig. 9(b), demonstrates the structural behavior by dividing the material distribution into two distinct zones. The lower region predominantly utilizes the red material, indicating areas subjected to high tensile forces. In contrast, the upper region employs the blue material, highlighting regions under significant compressive forces. This material placement enhances structural efficiency by assigning suitable materials to their respective mechanical roles.

Fig. 9(c) presents the outcome of the multi-material topology optimization, where four types of materials are strategically distributed to maximize load-bearing efficiency. The red region signifies the zones experiencing the highest tensile forces, followed by the yellow material, which facilitates the transmission of tensile stress from the red regions. The blue region indicates the highest compressive force zones, while the cyan material supports and extends the compressive functionality of the blue region.

Table 2 Result of the simple beam

		Materials volume fractions				Compliance	Cost(Bath)	Mass(Kg)
		steel	Iron	UHPC	Concrete			
	Price(Bath/t)	25000	20000	15000	2200			
	Young's modulus(Gpa)	200	100	55	28			
	Poisson's ratio	0.2	0.2	0.2	0.2			
	Density(kg/m³)	7850	6600	2300	2360			
Design 1	Steel	0.5	-	-	-	0.146369035	98125	3925
Design 2	Steel+UHPC	0.25	-	0.25	-	0.460009259	57687.5	2537.5
Design 3	Steel+Iron+UHCP+Concrete	0.15	0.1	0.15	0.1	0.97400773	48331.7	2418.5

Table 3 Percentage reduction in total cost and structural mass of multi-material designs compared to the single-material baseline.

	Cost Reduction(%)	Mass Reduction(%)
Design 1	-	-
Design 2	-41.21019108	-35.35031847
Design 3	-50.74476433	-38.38216561

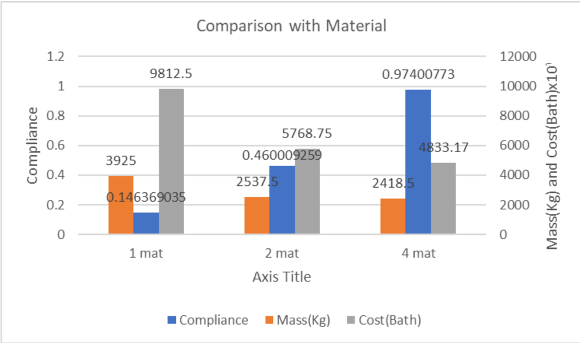

Fig. 10 Comparison of compliance, mass, and cost a) comparison of compliance b) comparison of mass c) comparison of cost

Table 2 and Fig. 10 present the compliance, mass, and cost of the designs in both tabular and graphical formats. The results show that although Design 1 (using a single material) achieves the lowest compliance value indicating the highest structural stiffness, it incurs significantly higher mass and cost. Specifically, the total mass is 3,925 kg and the total cost is 98,125 baht, the highest among the three designs.

In contrast, the use of multiple materials in Designs 2 and 3 leads to increased compliance (i.e., greater flexibility), but significantly reduces both mass and cost. As shown in Table 3, Design 2, which uses two materials (Steel and UHPC), reduces the structural mass by approximately 35.35% and the cost by 41.21% compared to Design 1. Although the compliance increases to 0.460 indicating reduced stiffness it remains within an acceptable range for applications where reduced weight and cost are prioritized.

Design 3, which utilizes four materials (Steel, Iron, UHPC, and Concrete), results in the highest compliance value (0.974), reflecting a notable reduction in structural strength. However, this design achieves the greatest economic benefit, with a 50.74% reduction in cost and a 38.38% reduction in mass compared to Design 1. Therefore, Design 3 may be considered suitable in cases where cost-efficiency outweighs the need for maximum structural strength.

5. Conclusions

In this research, applied the Automatic Image-Based SBFE-BESO method to analyze composite structures composed of more than one type of material (multi-material systems). The first invariant of the stress tensor was employed as a criterion to classify whether an element is subjected to tensile or compressive stress. Based on this classification, materials were divided into two groups: one suitable for tensile regions and the other for compressive regions.

The assignment of materials with varying stiffness levels was determined using sensitivity number values, which allowed the method to flexibly adjust the material distribution. The experimental results demonstrated that the proposed approach effectively reduced overall material usage while maintaining structural performance at a satisfactory level. Specifically, the use of multi-material systems enabled reductions in the volume of high-performance materials, leading to decreased structural mass, cost, and more efficient resource utilization.

Although there was a moderate reduction in structural strength, the resulting designs remained within acceptable limits, aligning with the objectives of the research. In conclusion, this research successfully extended the application of methodologies such as those in [2] and [10], demonstrating the capability of the SBFE-BESO approach in optimizing composite structures and achieving the desired outcomes.

Acknowledgement

Thank you for your cooperation in following the writing regulation strictly.

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